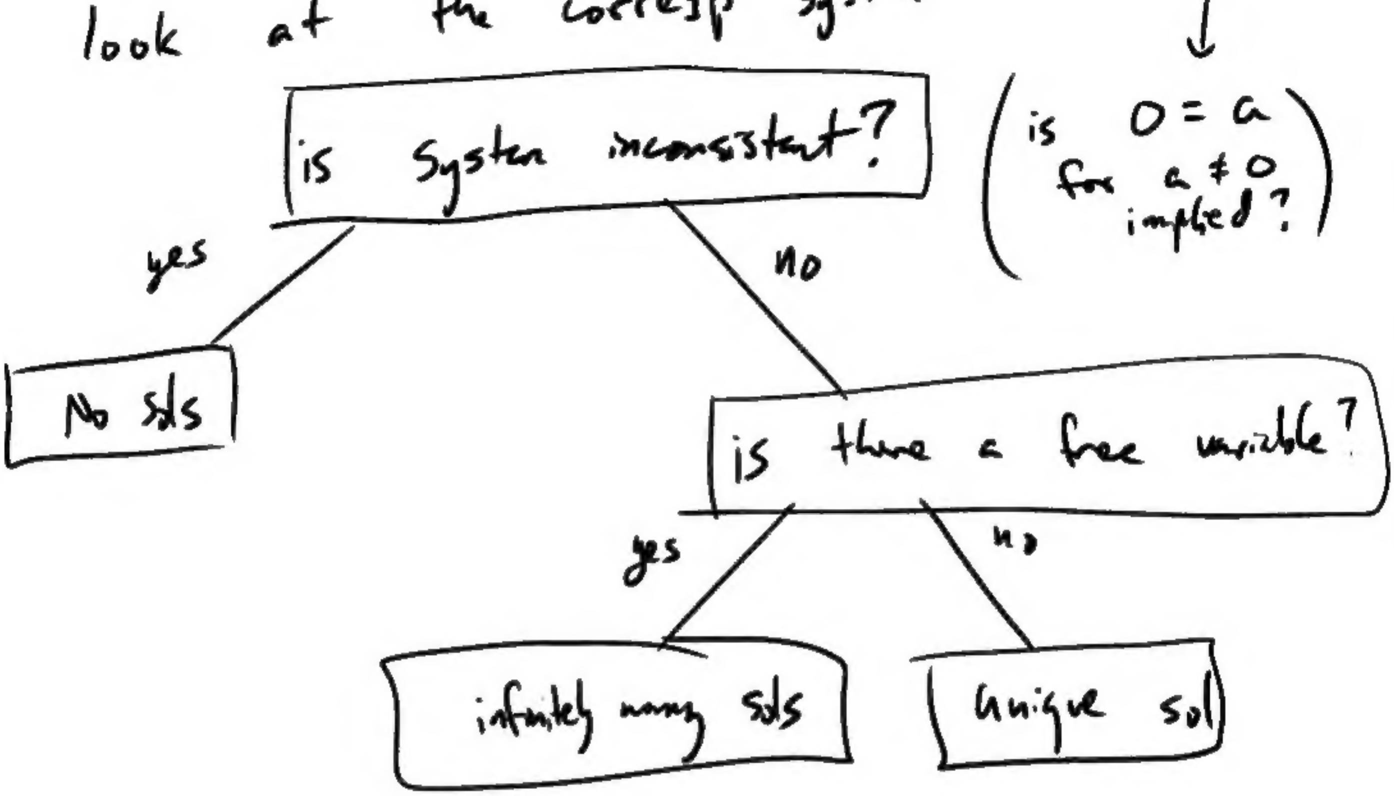
Trick (for number of solutions):

Take RREF $\rightsquigarrow [M|b]$

look at the corresp system



Last time: RREF and Consequences...
→ briefly defined and gave examples of
linear maps / linear functions / linear homomorphisms

Refresher on Functions.

Def^n: A function $f : S \to T$ is a rule of assignment, i.e. a method of assigning to each element of set S a unique member of set T.

Set = collection of objects

object in the set = element = member

The domain of $f: S \to T$ is denoted $\mathrm{dom}(f) = S$. The codomain of f is $\mathrm{cod}(f) = T$.

Ex: Calculus 1 is all about functions of the form $f: \mathbb{R} \to \mathbb{R}$.

ex: $f(x) = x^2$ w/ $\mathrm{dom}(f) = \mathbb{R}$ and $\mathrm{cod}(f) = \mathbb{R}$.

ex: $g(x) = x^2$ w/ $\mathrm{dom}(g) = \mathbb{R}$ and $\mathrm{cod}(g) = \mathbb{R}_{\geq 0}$

Ex: $L: \mathbb{R}^2 \to \mathbb{R}^1$ w/ $L\begin{bmatrix} x \\ y \end{bmatrix} = x + y$.

has domain $\mathbb{R}^2$ and codomain $\mathbb{R}^1$.

Non-Ex: "Food eaten today": People $\to$ foods is not a function, even though it is a rule of assignment (non-unique outputs)...

Non-Ex: $y = \pm\sqrt{1 - x^2}$ describes a circle in $\mathbb{R}^2$, but it is NOT a function because some input x (e.g. $x = 0$) has two associated output values.

Def^n: A linear map is a function $L: \mathbb{R}^n \to \mathbb{R}^m$ satisfying for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all $a \in \mathbb{R}$

① $L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y})$ ② $L(a\vec{x}) = aL(\vec{x})$.

NB: the definition from last time is equivalent to this one (i.e. any map satisfying that condition satisfies the new one and vice versa).

Prop: Suppose $L: \mathbb{R}^n \to \mathbb{R}^m$ is a function. The following are equivalent:

① for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all $a \in \mathbb{R}$ we have both

$$L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y}) \quad \text{and} \quad L(a\vec{x}) = aL(\vec{x}).$$

② for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all $a \in \mathbb{R}$ we have

$$L(\vec{x} + a\vec{y}) = L(\vec{x}) + aL(\vec{y}).$$

Lem: Linear maps in either sense always map the zero vector to the zero vector.

pf(Lem): Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a function.

① Assume $L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y})$ and $L(a\vec{x}) = aL(\vec{x})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all $a \in \mathbb{R}$.

Then $L(\vec{0}) = L(0 \cdot \vec{0}) = 0\, L(\vec{0}) = \vec{0}$.

② Assume $L(\vec{x} + a\vec{y}) = L(\vec{x}) + aL(\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $a \in \mathbb{R}$.

$$L(\vec{0}) = L(\vec{0} + (-1)\cdot\vec{0}) = L(\vec{0}) - 1L(\vec{0}) = \vec{0}.$$

Hence $L(\vec{0}) = \vec{0}$ in either case. $\boxtimes$

pf (of Proposition): Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a function

① ⇒ ②: Assume $L(\vec{x}+\vec{y}) \overset{*}{=} L(\vec{x}) + L(\vec{y})$ and $L(a\vec{x}) = aL(\vec{x})$ for all $\vec{x},\vec{y} \in \mathbb{R}^n$ and all $a \in \mathbb{R}$.

Thus
$$L(\vec{x} + a\vec{y}) = L(\vec{x} + (a\vec{y}))$$
$$= L(\vec{x}) + L(a\vec{y})$$
$$= L(\vec{x}) + aL(\vec{y})$$

So L satisfies the second condition as well.

② ⇒ ①: Assume $L(\vec{x} + a\vec{y}) = L(\vec{x}) + aL(\vec{y})$ for all $\vec{x},\vec{y} \in \mathbb{R}^n$ and all $a \in \mathbb{R}$. Now

$$L(\vec{x}+\vec{y}) = L(\vec{x} + 1\cdot\vec{y}) = L(\vec{x}) + 1L(\vec{y}) = L(\vec{x}) + L(\vec{y}).$$
$$L(a\vec{x}) = L(\vec{0} + a\vec{x}) = L(\vec{0}) + aL(\vec{x})$$
$$= \vec{0} + aL(x) = aL(\vec{x})$$

So L satisfies the first condition as well. $\boxtimes$

Ex: Let $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. I claim $L: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $L(\vec{x}) = M\vec{x}$ is a linear map.

Indeed, supposing $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and $a \in \mathbb{R}$:

$$L(\vec{x}+\vec{y}) = M(\vec{x}+\vec{y}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + x_2 + y_2 \\ x_2 + y_2 \end{bmatrix}$$

Same output!

and
$$L(\vec{x}) + L(\vec{y}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 + y_2 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_2 + y_1 + y_2 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + x_2 + y_2 \\ x_2 + y_2 \end{bmatrix}$$

So we have $L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y})$ in this case.

$$L(a\vec{x}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\left(a\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} ax_1 \\ ax_2 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + ax_2 \\ ax_2 \end{bmatrix} = \begin{bmatrix} a(x_1 + x_2) \\ ax_2 \end{bmatrix} = a\begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix} = aL(\vec{x})$$

Thus, L is a linear map! $\boxed{//}$.

Let M be an $m \times n$ matrix. Then M determines a linear map $L_M : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ via $L_M(\vec{x}) = M\vec{x}$.

Point: Matrices give linear maps ☺.

Ex: Let $M = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$. The associated linear map:

① has domain $\mathbb{R}^3$

② has codomain $\mathbb{R}^2$

$$L_M(\vec{x}) = M\vec{x} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + 2x_2 + x_3 \\ -x_1 + x_2 + 3x_3 \end{bmatrix} \leftarrow 2 \times 1$$

$$= \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ 3x_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

So in this example, L_M takes each vector $\vec{x}$ to a linear combination of the columns of M... This happens in General!

<u>Prop</u>: If $M = [\vec{c}_1 | \vec{c}_2 | \cdots | \vec{c}_n]$ has columns $\vec{c}_1, \vec{c}_2, \cdots, \vec{c}_n$, then the linear map $L_M : \mathbb{R}^n \to \mathbb{R}^m$ has formula

$$L_M \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{c}_1 + x_2 \vec{c}_2 + \cdots + \vec{x}_n \vec{c}_n .$$

In particular, every range-value of L_M is a linear combination of the columns of M.

<u>Ex</u>: Write the range values of L_M as a linear combination of vectors for matrix

$$M = \begin{bmatrix} -1 & -2 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 3 & 0 \end{bmatrix}.$$

Note $L_M : \mathbb{R}^3 \to \mathbb{R}^4$ as a function. Moreover

$$L_M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Hence $\underline{\text{range}}(L_M) = \left\{ s\begin{bmatrix}-1\\3\\0\\1\end{bmatrix} + t\begin{bmatrix}-2\\1\\0\\3\end{bmatrix} + u\begin{bmatrix}1\\1\\0\\0\end{bmatrix} : s,t,u \in \mathbb{R} \right\}$

<u>NB</u>: the <u>range</u> of function $f : S \to T$ is

$$\text{range}(f) := \{ t : t = f(s) \text{ for some } s \in S \}$$

i.e. $\text{range}(f) = \{ f(s) : s \in \text{dom}(f) \}$.

<u>NB</u>: I keep saying "if L is determined by a matrix..."
Actually, <u>every</u> linear map is determined by a matrix...
↳ proof coming soon (but not too soon :)).

Back to linear systems: represents
If $[M|\vec{b}]$ is a linear system, then the solutions of the system satisfy $M\vec{x} = \vec{b}$. i.e.

$L_M(\vec{x}) = M\vec{x} = \vec{b}$, so $[M|\vec{b}]$ has a solution if and only if $\vec{b} \in \text{range}(L_M)$.

in other words, $\vec{b}$ is a linear combination of the columns of M... i.e. range elements of L_M correspond to solvable linear systems with matrix of coefficients M. :)

Ex: Is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in the range of $L(\vec{x}) = \begin{bmatrix} 3x - y + z \\ -x + y + z \end{bmatrix}$?

Sol: $L\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \iff \begin{bmatrix} 3x & -y & +z \\ -x & +y & +z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\iff \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\iff \left[\begin{array}{ccc|c} 3 & -1 & 1 & 2 \\ -1 & 1 & 1 & 1 \end{array}\right]$ has a solution...

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 2 \\ -1 & 1 & 1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 0 & 2 & 4 & 5 \\ -1 & 1 & 1 & 1 \end{array}\right]$$

$\rightsquigarrow \Big[$ Finish for homework